A simplified force equation for coaxial cylindrical magnets and thin coils

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***Abstract*—A recently-published equation for calculating the force between coaxial cylindrical magnets is presented in simplified form. The revised equation is now very compact: it is defined in terms of fewer parameters and contains fewer terms than the original equation. The new equation is purely real, unlike the original which contained imaginary components. As a result of the simplifications, the new equation is demonstrably faster to evaluate than the original, improving its utility for parametric optimisation. A reference implementation is provided for Matlab and Mathematica.**

I. Introduction

Ravaud et al. [1] recently published an expression for the forces between two cylindrical magnets or thin coils (which are equivalent electromagnetically). In this paper, we will present a simplification of their equation. This simplification results in a faster execution time and more convenient calculation with numerical software.

A reference implementation of the equations derived in this paper is available (http://github.com/wspr/magcode) for both Matlab and Mathematica; see respectively the files 'ravaud-cylmag.m' and 'ravaud-cylmag.nb' in the 'examples/' directory. The graphs presented in this paper have been generated with these files; this work is therefore reproducible and may be used by others without problems stemming from implementation details.

In Section II, we will briefly discuss previous work in this area; Section III defines the geometry of the system under investigation; Section IV contains the simplifications and the presentation of the new equation; and Section V contains comments on the numerical evaluation of the equation with an example implementation of the previous and the new equation to demonstrate their equivalence.

II. Background

Although an analytical form of the force between cuboid magnets has been known for some time [2], a similar closed-form solution for the force between cylindrical magnets has proved more difficult to obtain. A fast and accurate model to calculate cylindrical magnetic forces is useful for optimising such devices as vibration isolation systems with magnetic springs for which coarse force-displacement approximations are often used [3], [4], [5]. Having an expression to calculate the force between cylindrical magnets also allows one to calculate the force between ring magnets using the principle of superposition; ring magnets are widely used for magnetic bearings, and their forces until recently have been modelled using semi-numerical evaluation of integral equations based on the Coulombian charge model [6].

In previous literature on modelling the forces between cylindrical magnets, Nagaraj [7] investigated and compared the force between cuboid and cylindrical magnets with arbitrary displacements using numerical integration to calculate his results; Furlani [8] calculated the force between radially-aligned ring magnets using a numerical discretisation of the magnet volume using theory developed in more detail in his book [9]. Hull et al. [10] presented integral equations for calculating the radial and axial forces between a cylindrical magnet and a superconductor, which is equivalent to the force between two cylindrical magnets, and Bassani [11] presented integral equations for calculating the radial and axial forces between ring magnets. Such integral equations require numerical methods to evaluate. Most recently, Ravaud et al. [1] derived a closed-form solution using elliptic integrals for the forces between radially-aligned cylindrical magnets; their result is the most straightforward method yet presented for calculating forces in this configuration.

The equation for the force between cylindrical magnets can also be used to calculate the force between thin coils with many axial turns. In related work, Kim et al. [12] presented a different integral equation for the radial force between (single-turn) circular coils with eccentric radial displacement, for which further application of their results is required to calculate the forces between coils with many turns, such as for the system examined here. An expression for the force between thin coaxial coils has also been published by Babic et al. [13]; it too is more complex than the expression to be presented in the current work.

III. Geometry

The system consists of two cylindrical magnets or current-carrying coils which have a relative axial displacement between them but are aligned radially, as shown in Figure 1.

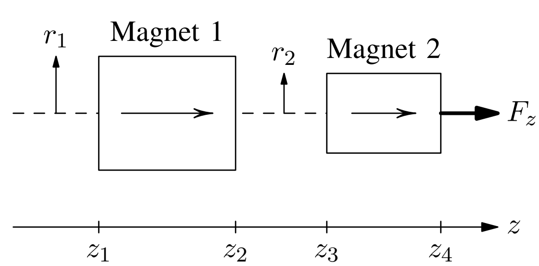

Fig. 1. Cross section of the system composed of coaxial cylindrical magnets or coils with a generate force on the second magnet (or coil). Axial displacement between the magnets may be positive or negative, and their volumes may even overlap in the case of a magnet located inside a coil.

The original equation of Ravaud et al. [1] defines the axial force F_z in terms of ten auxiliary parameters (ω, τ, ϵ, ν, ι, β, κ, γ, δ, ψ) which are defined in terms of the geometric constants of the system [1, Table III]. With a recasting of the parameters in the original paper, the new equations presented in this paper are defined using the following five terms:

$$\begin{aligned} c_1 &= z_i - z_j, & c_4 &= \sqrt{c_1^2 + c_2^2}, \\ c_2 &= r_1 - r_2, & & \\ c_3 &= r_1 + r_2, & c_5 &= \frac{\sqrt{c_1^2 + c_3^2}}{c_4}, \end{aligned} \tag{1}$$

where r_1 and r_2 are the magnet/coil radii and $z_1, \ldots, z_4$ are the axial positions of the magnet/coil faces (see Figure 1). Indices i and j refer to the occurence of c_1 within a double summation (see equation (4) for example). The magnets have magnetisation J_1 and J_2 respectively in the positive axial direction; that is, for positive J_1 and J_2 the magnets are in attraction.

The geometry and force equations in this paper are defined to calculate the force on the second magnet/coil due to the first; the sign may be reversed to obtain the force on the first magnet/coil due to the second.

IV. Simplification of the force expression

In this section, we will present the simplifications made to the equation of Ravaud et al. [1, Eqs (22)–(24)], which we refer to as the 'original equation' herein.

A. Incomplete to complete elliptic integrals

In this paper, the elliptic integrals are defined in terms of their parameter m rather than their modulus $k = \sqrt{m}$. Both representations are equivalent, but it is more common for numerical methods (such as found in Mathematica and Matlab) to use the parameter form instead of the modulus form.

The incomplete elliptic integrals in the original equation can be transformed into complete elliptic integrals by using the following recipocal-modulus transformations [14, adapted below in Appendix A], where $F(\phi \,|\, m)$ and $E(\phi \,|\, m)$ are the incomplete integrals of the first and second kind, respectively:

$$F\left(\arcsin\left(\sqrt{\frac{1}{m}}\right) \middle| \, m\right) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right), \tag{2}$$

$$\begin{aligned} E\left(\arcsin\left(\sqrt{\frac{1}{m}}\right) \middle| \, m\right) &= \sqrt{m} E\left(\frac{1}{m}\right) \\ &\quad + \left[\frac{1}{\sqrt{m}} - \sqrt{m}\right] K\left(\frac{1}{m}\right). \end{aligned} \tag{3}$$

Using the complete forms of the elliptic integrals leads to faster execution and a simpler expression. It is more common to find numerical software to calculate the complete elliptic integrals but not the incomplete ones.

B. Initial replication of the original equation

Using the parameters (1) and rewriting and simplifying the original equation using the transformations (2)–(3), the axial force F_z between two cylindrical magnets or coils is given by

$$F_z = \frac{J_1 J_2}{4\mu_0} \sum_{i=1}^{2} \sum_{j=3}^{4} c_1 c_4 f_z [-1]^{i+j}, \tag{4}$$

where J_1 and J_2 are the magnetisations of the magnets in the positive z direction, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{NA}^{-2}$ is the magnetic constant, and f_z is an intermediate function given by

$$\begin{aligned} f_z &= \mathrm{i}\left[c_5^2 - 1\right]\left[K\big(c_5^2\big) + \frac{3}{\sqrt{c_5}} K\left(\frac{1}{c_5^2}\right)\right] \\ &\quad + 2\mathrm{i}\left[E\big(c_5^2\big) - c_5 E\left(\frac{1}{c_5^2}\right)\right] + \frac{c_2^2 + c_3^2}{c_4^2} K\big(1 - c_5^2\big) \\ &\quad - 2\left[\frac{c_3}{c_4}\right]^2 \Pi\left(1 - \left[\frac{c_3}{c_2}\right]^2 \middle| \, 1 - c_5^2\right), \end{aligned} \tag{5}$$

where $\mathrm{i} = \sqrt{-1}$ and $K(m)$, $E(m)$, and $\Pi(n \,|\, m)$ are the complete elliptic integrals of the first, second, and third kind, respectively.

The simplified equation (5) has been re-derived from the original derivation of Ravaud et al. [1] and hence differs in form from their published equation.

C. Removal of zero-valued elliptic integral terms

Further simplification of equation (5) is possible. This expression for f_z, like the original, returns complex values but only the real component $\Re(f_z)$ is used for the force calculation; any imaginary terms are simply ignored. Consider parameters (1); since $c_3^2 > c_2^2$ it follows that $c_5^2 > 1$ and $0 < 1/c_5^2 < 1$. But the elliptic integral $K(m)$ is purely real for $0 \leq m \leq 1$ and hence $\Re\big(\mathrm{i}K\big(1/c_5^2\big)\big)$ is always zero. Similarly for the elliptic integral of the second kind, such that $\Re\big(\mathrm{i}E\big(1/c_5^2\big)\big) = 0$. Therefore equation (5) can be reduced by eliminating the elliptic integrals with parameter $1/c_5^2$ to obtain the expression

$$\begin{aligned} f_z &= \mathrm{i}\left[c_5^2 - 1\right] K\big(c_5^2\big) + 2\mathrm{i}E\big(c_5^2\big) + \frac{c_2^2 + c_3^2}{c_4^2} K\big(1 - c_5^2\big) \\ &\quad - 2\left[\frac{c_3}{c_4}\right]^2 \Pi\left(1 - \left[\frac{c_3}{c_2}\right]^2 \middle| \, 1 - c_5^2\right). \end{aligned} \tag{6}$$

D. From complex-valued to purely real output

Equation (6) can still be transformed into a more convenient form by eliminating the complex components of the equation in order to make it purely real. The complex expressions for $K\big(c_5^2\big)$ and $E\big(c_5^2\big)$ can be rewritten in real and imaginary parts based on the following identities [14, §19.7.3] which hold for $m > 1$:

$$K(m) = \frac{1}{\sqrt{m}}\left[K\left(\frac{1}{m}\right) - \mathrm{i}K\left(1 - \frac{1}{m}\right)\right], \tag{7}$$

$$E(m) = \sqrt{m}\left[E\left(\frac{1}{m}\right) - \left[1-\frac{1}{m}\right]K\left(\frac{1}{m}\right)\right] + \mathrm{i}\left[\sqrt{m}\,E\left(1-\frac{1}{m}\right) - \frac{1}{\sqrt{m}}K\left(1-\frac{1}{m}\right)\right]. \tag{8}$$

Using equations (7) and (8), the real components of the complex terms in equation (6) can be expressed directly as

$$\Re\left(\mathrm{i}\left[c_5^2-1\right]K\left(c_5^2\right) + 2\mathrm{i}E\left(c_5^2\right)\right) = \left[c_5+\frac{1}{c_5}\right]K\left(1-\frac{1}{c_5^2}\right) - 2c_5\,E\left(1-\frac{1}{c_5^2}\right). \tag{9}$$

The now purely-real expression for f_z in simplified form is

$$f_z = \left[c_5+\frac{1}{c_5}\right]K\left(1-\frac{1}{c_5^2}\right) - 2c_5 E\left(1-\frac{1}{c_5^2}\right) + \frac{c_2^2+c_3^2}{c_4^2}K\left(1-c_5^2\right) - 2\left[\frac{c_3}{c_4}\right]^2\Pi\left(1-\left[\frac{c_3}{c_2}\right]^2 \middle|\, 1-c_5^2\right). \tag{10}$$

E. Elliptic integrals with negative parameter

The latter two terms in equation (10) cannot easily be calculated in some mathematical software (for example, at time of writing, Matlab), as many numerical solutions for the elliptic integrals assume input parameters between zero and unity, whereas $1 - c_5^2 < 0$. (The transformation of equation (9) avoided a similar problem for the first two terms of equation (10).) For the complete elliptic integral of the first kind, the imaginary-modulus transformation (see equation (25) in Appendix B) is used to convert it into a more easily calculable form:

$$K(m) = \frac{1}{\sqrt{1-m}}K\left(\frac{m}{m-1}\right), \quad m < 0\,. \tag{11}$$

For the complete elliptic integral of the third kind, the following transformation can be used to calculate values for $\Pi(n\,|\,m)$ when $m < 0$ and $n < 0$ (Appendix B, equation (29)):

$$\Pi(n\,|\,m) = \frac{1}{[m-n]\sqrt{1-m}}\left[m\,K\left(\frac{m}{m-1}\right) - n\,\Pi\left(\frac{m-n}{m-1}\,\middle|\,\frac{m}{m-1}\right)\right]. \tag{12}$$

Applying the imaginary-modulus transformations (11) and (12) simplifies equation (10) yet further as the two elliptic integrals of the first kind now take the same parameter $1-1/c_5^2$. We now have for the final simplified expression

$$F_z = \frac{J_1 J_2}{2\mu_0}\sum_{i=1}^{2}\sum_{j=3}^{4} a_1 a_2 a_3 f_z' \,[-1]^{i+j}\,, \tag{13}$$

where the intermediate expression is

$$f_z' = K(a_4) - \frac{1}{a_2}E(a_4) + \left[\frac{a_1^2}{a_3^2}-1\right]\Pi\left(\frac{a_4}{1-a_2}\,\middle|\, a_4\right), \tag{14}$$

where the redefined parameters are

$$a_1 = z_i - z_j, \tag{15}$$

$$a_2 = \frac{[r_1-r_2]^2}{a_1^2} + 1, \tag{16}$$

$$a_3 = \sqrt{[r_1+r_2]^2 + a_1^2}, \qquad \text{or } a_3 = \sqrt{\frac{4r_1r_2}{a_4}}, \tag{17}$$

$$a_4 = \frac{4r_1r_2}{[r_1+r_2]^2 + a_1^2}, \qquad 0 < a_4 \le 1. \tag{18}$$

As a result of the simplifications presented here, equation (13) contains one-third the number of terms compared to the original equation (three instead of nine) and is defined in terms of four instead of ten parameters. Additionally, the complete elliptic integrals of the first, second, and third kind can all be calculated simultaneously with a single iteration of the arithmetic-geometric mean approach [14, §19.8(i)], as they all take the same parameter a_4. This makes equation (14) particularly efficient to implement.

V. Numerical evaluation of the axial force

A. Singularities

Numerical singularities occur when an expression is mathematically continuous but terms within the expression approach infinity; care must be taken when evaluating such expressions numerically. There are two numerical singularities in equation (13). The first occurs when the radii are equal such that $a_2 = 1$ and the following term disappears as $\Pi(\pm\infty\,|\,m) = 0$:

$$\left[\frac{a_1^2}{a_3^2}-1\right]\Pi\left(\frac{a_4}{1-a_2}\,\middle|\,a_4\right) = 0, \quad a_2 = 1\,. \tag{19}$$

The second numerical singularity occurs when the magnets/coils have coincident faces such that $a_1 = 0$ for some values of i and j in the double summation. In this case, new parameter a_2 contains the coefficient $1/a_1^2 = 1/0$. This singularity can be avoided entirely since the coincident faces generate no component of the force between them, and hence the entire intermediate expression within the summation $a_1a_2a_3f_z'$ can be defined as zero when $a_1 = 0$.

B. Implementation efficiency

Evaluated in Mathematica (including branching to avoid singularities), the new equation (13) took 0.26 ms on a notebook computer to calculate the force at a single location (10000 averages with random input variables). The original equation in the same configuration took 2.2 ms to evaluate, which is over eight times slower than for the new equation. For researchers performing design optimisations with variations over a large number of parameters, such improvements are useful in minimising the total computation time of the optimisation process.

C. Example

The equivalence of the simplified equation is demonstrated with an example in which a magnet of length $h_2 = z_4 - z_3$ is located axially centred within a coil of length $h_1 = z_2 - z_1$ and

r_1	20 mm
h_1	20 mm
J_1	$\mu_0 NI/h_1$
N	100
I	1 A
r_2	15 mm
h_2	15 mm
J_2	1 T

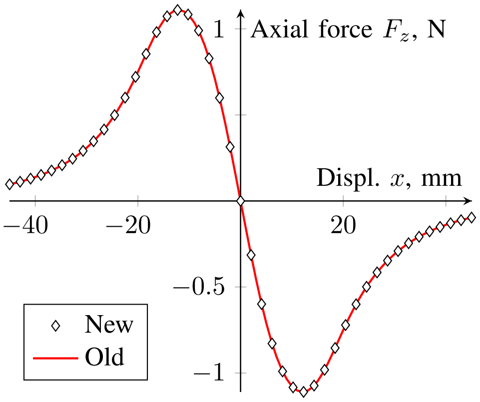

Fig. 2. An example of the force generated on a cylindrical magnet by a coil. Displacement is defined as zero when the coil and magnet are axially centred with respect to each other. The new equation is implemented in Matlab vs the old equation in Mathematica to ensure the two are numerically equivalent.

displaced in equal amounts in the positive and negative axial directions. That is, axial displacement x between the centres of the magnet and the coil is defined as

$$x = \tfrac{1}{2}\left[z_3 + z_4\right] - \tfrac{1}{2}\left[z_1 + z_2\right]. \tag{20}$$

Figure 2 shows the parameters used in the simulations and displays the respective outputs calculated with the old equation using Mathematica and the new equation using Matlab. These simulations are performed in the example files mentioned in Section I.

In the simulation results, the magnet experiences zero force when it is axially centred inside the coil. With axial displacement, a restoring force is applied by the coil in the opposite direction of displacement. As shown in Figure 2, the results are identical for both the original and the simplified equations.

VI. Conclusion

We have presented a simplification of Ravaud et al.'s [1] closed-form equation for calculating the force between radially-aligned cylindrical magnets. The simplified equation is defined with fewer than half the number of parameters, contains one-third of the terms, and its output is purely real whereas the original equation evaluated to a complex value of which the imaginary component was ignored. Confirmation has been shown that the new expression gives the same results and is almost an order of magnitude faster due to its simpler form. This new expression has been implemented in Matlab and Mathematica and the code is freely available for use by the general public.

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Appendix

A. Reciprocal-modulus transformations

The reciprocal-modulus transformations are used to cast an elliptic integral with parameter $m > 1$ into a form with $0 \le m \le 1$ for more convenient numerical evaluation.

The following material is adapted from Ref. [14, §19.7.4]. Note that the elliptic integrals in Ref. [14] are defined in terms of their modulus $k = \sqrt{m}$, contrary to the terminology used in this paper. Defining the relation when $m > 1$,

$$\sqrt{m}\sin\beta = \sin\phi\,, \tag{21}$$

the reciprocal-modulus transformation for the elliptic integral of the first kind is

$$F\left(\phi \,\middle|\, \frac{1}{m}\right) = \sqrt{m}F(\beta\,|\,m), \tag{22}$$

or more directly, in terms of parameter $m > 1$ and phase β,

$$F(\beta \,|\, m) = \frac{1}{\sqrt{m}} F\left(\arcsin(\sqrt{m} \sin\beta) \,\middle|\, \frac{1}{m} \right). \tag{23}$$

The second elliptic integral has a similar transformation,

$$E\left(\phi \,\middle|\, \frac{1}{m}\right) = \frac{1}{\sqrt{m}} \left[E(\beta \,|\, m) - [1-m] \, F(\beta \,|\, m) \right], \tag{24}$$

or, again given $m > 1$ and some β,

$$E(\beta \,|\, m) = \left[\frac{1}{\sqrt{m}} - \sqrt{m} \right] F\left(\arcsin\left(\sqrt{m} \sin\beta\right) \,\middle|\, \frac{1}{m} \right) + \sqrt{m} E\left(\arcsin(\sqrt{m} \sin\beta) \,\middle|\, \frac{1}{m} \right).$$

B. Imaginary-modulus transformations

The following material is adapted from Ref. [14, §19.7.5]. The imaginary-modulus transformations are used to transform elliptic integrals with negative parameter (i.e., when modulus $k = \sqrt{m}$ is imaginary) into a form with $0 \le m \le 1$. For the first elliptic integral with $m > 0$:

$$F(\phi \,|\, {-m}) = \frac{1}{\sqrt{1+m}} F\left(\theta \,\middle|\, \frac{m}{1+m}\right), \tag{25}$$

where θ is defined by

$$\sin\theta = \frac{\sin\phi \sqrt{1+m}}{\sqrt{1+m\sin^2\phi}}. \tag{26}$$

When looking at the complete elliptic integrals, $\phi = \pi/2$ and the imaginary-modulus transformation (25) reduces to Equation (11).

The imaginary-modulus transformation for the second elliptic integral is included for completeness, again for $m > 0$:

$$E(\phi \,|\, {-m}) = -\frac{m \sin\theta \cos\theta}{\sqrt{1+m\cos^2\theta}} + \sqrt{1+m}\, E\left(\theta \,\middle|\, \frac{m}{1+m}\right). \tag{27}$$

Substituting $\phi = \pi/2$ into equation (27) produces the simpler transformation for the complete elliptic integral of the second kind:

$$E(-m) = \sqrt{1+m}\, E\left(\frac{m}{1+m}\right). \tag{28}$$

Finally for the elliptic integral of the third kind the imaginary-modulus transformation for $m > 0$ is:

$$\Pi(n;\, \phi \,|\, {-m}) = \frac{1}{[n+m]\sqrt{1+m}} \left[mF\left(\theta \,\middle|\, \frac{m}{1+m}\right) + n\Pi\left(\frac{n+m}{1+m};\, \theta \,\middle|\, \frac{m}{1+m} \right) \right], \tag{29}$$

which reduces to Equation (12) when $\phi = \pi/2$ in the complete form.